M.Sc. (Mathematics) (New CBCS Pattern) Semester-I

PSCMTH01: - Group Theory & Ring Theory

P. Pages : 2

Time : Three Hours

GUG/W/24/13737

Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) Let N be a subgroup of Group G. Then prove that the following are equivalent: **10**

i) $N \triangleleft G$.

ii) $xNx^{-1} = N$ for every $x \in G$.

iii) $xN = Nx$ for every $x \in G$.

iv) $(xN)(yN) = xyN$ for every $x, y \in G$.

b) State and prove first isomorphism theorem. **10**

OR

c) Let G be a group. Then prove that: **10**

i) The set of conjugate classes of G is a partition of G.

ii) $|C(a)| = [G : N(a)]$.

d) Prove that the set $Aut(G)$ of all automorphism of a group G is a group under composition of mappings, and $ln(G) \triangleleft Aut(G)$. Moreover, show that $G / Z(G) \cong In(G)$. **10**

UNIT – II

2. a) Prove that every finite group has a composition series. **10**

b) Prove that an abelian group G has a composition series if and only if G is finite. **10**

OR

c) Prove that: A group G is solvable group if and only if G has a normal series with abelian factors. Further, a finite group is solvable if and only if its composition factors are cyclic groups of prime orders. **10**

d) Prove that: If G be a nilpotent group, then every subgroup of G and every homomorphic image of G are nilpotent. **10**

UNIT – III

3. a) Show that : If each element $\neq e$ of a finite group G is of order 2, then $|G| = 2^n$ and $G \cong C_1 \times C_2 \times .... \times C_n$, where C_i are cyclic and $|C_i| = 2$. **10**

b) Let G be a finite group, and let p be a prime. If p^m divides |G|, then prove that G has a subgroup of order p^m. **10**

OR

c) Prove that : If G is a finite group and p a prime, then all Sylow p-subgroups of G are conjugate, and their number n_p divides $_o(G)$ and satisfies $n_p \equiv 1(\mathrm{mod}\, p)$. **10**

d) Let G be a group of order 108. Show that there exists a normal subgroup of order 27 or 9. **10**

UNIT – IV

4. a) For any ring R and any ideal $A \neq R$, then prove that the following are equivalent: **10**
   i) A is maximal
   ii) The quotient ring R/A has no nontrivial ideals.
   iii) For any element $x \in R, x \notin A, A + (x) = R$.

b) Prove that: In a nonzero commutative ring with unity, an ideal M is maximal if and only if R/M is a field. **10**

OR

c) Let $f : R \to S$ be a homomorphism of a ring R into a ring S with kernel N. Then prove that $R / N \cong \mathrm{Imf}$. **10**

d) Let R be a commutative principal ideal domain with identity. Then prove that any non-zero ideal $P \neq R$ is prime if and only if it is maximal. **10**

5. a) Define: **5**
   i) Maximal normal subgroup.
   ii) Simple group.

b) Prove that a group of order p^n (Prime) is nilpotent. **5**

c) Prove that: A finite group G is a p-group iff its order is a power of p. **5**

d) Let $f : R \to S$ be a homomorphism of a ring R into a ring S. Then prove that $\mathrm{Kerf} = (0)$ if and only if f is 1-1. **5**

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