M.Sc. - I (Mathematics) (NEP Pattern) Semester-I

NEP-63 - Major (DSC) Paper-III - Linear Algebra

P. Pages : 2 **GUG/W/24/15114**

Time : Three Hours Max. Marks : 80

Notes :
1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT-I

1. a) Let V be a finite dimensional vector space over K, and let X and Y be finite subsets of V. If Y is linearly independent and $V = \langle X \rangle$, then prove that $|Y| \le |X|$. **8**

b) Let V and V' be finite dimensional vector spaces over K. Then prove that $V \simeq V'$ if and only if $\dim V = \dim V'$. **8**

OR

c) If W is a subspace of a vector space V. Then prove that **8**

i) for $x, y \in V$, $x + W = y + W$ if and only if $x - y \in W$.

ii) any two cosets of W in V are either identical or disjoint.

d) Let V be a finite dimensional vector space over K, and let W be a subspace of V. Then prove that $\dim V = \dim W + \dim V / W$. **8**

UNIT-II

2. a) Find eigenvalues and eigenvectors for the matrix. **8**

$$\begin{bmatrix} 6 & -1 & 2 \\ 4 & 1 & 2 \\ -10 & 0 & 3 \end{bmatrix}$$

b) Prove that the geometric multiplicity of an eigenvalue of a linear operator can not exceed its algebraic multiplicity. **8**

OR

c) Let V be a finite dimensional vector space over K of dimension n and let T be a linear operator on V. If $m_T(x) = p(x)^r$, where $p(x)$ is a monic irreducible polynomial of degree m, then prove that m divides n. **8**

d) Let V be a finite dimensional vector space over K and let T be a linear operator on V. Then prove that V is a direct sum of T-cyclic subspaces. **8**

UNIT-III

3. a) Let W be a subspace of finite dimensional inner product space V and let $v \in V$. Then prove that **8**

i) $\|v - Pr_W(v)\| \le \|v - w\|$ for all $w \in W$. The equality holds if and only if $w = Pr_W(v)$.

ii) If $\{x_1, x_2, \ldots\ldots, x_m\}$ is an orthonormal basis of W, then $Pr_W(v) = \sum_{i=1}^{m} (v, x_i)\, x_i$.

b) Use Gram-Schmidt orthonormalization process to obtain an orthonormal basis spanned by vectors $\begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ in the standard inner product space. **8**

OR

c) Let V and W be finite dimensional inner product spaces over F. Then prove that **8**

i) If $S, T \in L(V, W)$, then $(S+T)^* = S^* + T^*$, and for $\lambda \in f$, $(\lambda S)^* = \bar{\lambda} S^*$.

ii) If $S \in L(V, W)$, then $S^{**} = S$, where $S^{**} = (S^*)^*$

iii) If $S, T \in L(V)$, then $(ST)^* = T^* S^*$

iv) If $T \in L(V)$ and T is invertible, then $(T^*)^{-1} = (T^{-1})^*$.

d) State and prove Schur's theorem. **8**

UNIT-IV

4. a) Let ϕ be a reflexive bilinear form on a finite dimensional vector space V over K. Then prove that ϕ is non degenerate if and only if the matrix of ϕ with respect to an ordered basis of V is invertible. **8**

b) Let A and B be invertible matrices of the same size. Prove that the following are equivalent: **8**

i) A and B are congruent

ii) A^{-1} and B^{-1} are congruent

iii) A^t and B^t are congruent

OR

c) State and prove Witt's theorem. **8**

d) Prove that a symmetric bilinear form on a finite dimensional vector space over field K of characteristic not equal to 2 is diagonalizable. **8**

5. a) Let V and V' be vector spaces over K, and let $T : V \to V'$ be injective linear transformation. If $x_1, ..., x_n$ are linearly independent elements of V, then prove that $T(x_1), ..., T(x_n)$ are linearly independent elements of V'. **4**

b) Let V be a finite dimensional vector space over K and let T be a linear operator on V. Then prove that a scalar λ in K is an eigenvalue of T if and only if $\det(T - \lambda I) = 0$. **4**

c) Prove that an orthogonal set of non zero vectors is linearly independent. **4**

d) Let ϕ be a non degenerate reflexive bilinear form on a finite dimensional vector space V over K. For a subspace S of V, prove that $S^{\perp\perp} = S$. **4**

****