M.Sc. (Mathematics) (New CBCS Pattern) Semester-II

PSCMTH06 : Field Theory

P. Pages : 2 **GUG/W/24/13746**

Time : Three Hours Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each question carries equal marks.

UNIT - I

1. a) Prove that an irreducible element in a commutative principal ideal domain is always prime. **10**

b) Let R be a UFD. If $f(x), g(x) \in R[x]$, then prove that $C(fg) = C(f) \cdot C(g)$. **10**

OR

c) Let R be UFD, and $a, b \in R$. Then prove that there exists a greatest common divisor of a & b that is uniquely determined to within an arbitrary unit factor. **10**

d) Prove that every PID is a UFD, but a UFD is not necessarily a PID. **10**

UNIT - II

2. a) Let $F(x) = a_0 + a_1 x + ... + a_n x^n \in \mathbb{Z}[x]$, $n \geq 1$. If there is a prime P such that $P^2 \nmid a_0$, $P \mid a_0, P \mid a_1, ..., P \nmid a_n$, then prove that F(x) is irreducible over Q. **10**

b) Let E be an algebraic extension of F, and let $\sigma : E \to E$ be an embedding of E into itself over F. Then prove that σ is onto map. **10**

OR

c) Let $F \subseteq E \subseteq K$ be fields. If $[K:E] < \infty$ and $[E:F] < \infty$, then Prove that **10**

i) $[K:F] < \infty$

ii) $[K:F] = [K:E].[E:F]$

d) Let E be an algebraic extension of a field F, and let $\sigma : F \to L$ be an embedding of F into an algebraically closed field L. Then prove that σ can be extended to an embedding $n : E \to L$. **10**

UNIT - III

3. a) Let K be a splitting field of the polynomial $F(x) \in F[x]$ over a field F. If E is another splitting field of $F(x)$ over F, then prove that there exists an isomorphism $\sigma : E \to K$ that is identity on F. **10**

b) If E is a finite separable extension of a field F, then prove that E is a simple extension of F. **10**

OR

c) Let E be an extension of a field F, & let $\propto \in E$ be algebraic over F. Then prove that $\propto$ is separable over F iff $F(\propto)$ is a separable extension of F. **10**

d) If the multiplicative group F^* of nonzero elements of a field F is cyclic, then prove that F is finite. **10**

UNIT - IV

4. a) If $F(x) \in F[x]$ has r distinct roots in its splitting. Field E over F, then prove that the Galois group $G(E/F)$ of F(x) is a subgroup of the symmetric group S_r. **10**

b) Prove that every polynomial $F(x) \in \mathbb{C}[x]$ factors into linear factors in $\mathbb{C}[x]$. **10**

OR

c) Prove that the Galois group of $x^3 - 2 \in Q[x]$ is the group of symmetries of the triangle. **10**

d) Prove that the Galois group of $x^4 + 1 \in Q[x]$ is the Klein four-group. **10**

5. a) Define : **5**

i) Unique factorization domain ii) Euclidean domain.

b) Let $F(x) \in F[x]$ be a polynomial of degree 2 or 3. Then prove that F(x) is reducible if & only if F(x) has a root in F. **5**

c) Define : **5**

i) Normal extension ii) Separable extension.

d) State the Fundamental Theorem of Galois Theory. **5**

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