M.Sc. (Mathematics) (New CBCS Pattern) Semester-II

PSCMTH07 : Lebesgue Measure Theory

P. Pages : 2 **GUG/W/24/13747**

Time : Three Hours * 5 4 2 4 * Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT - I

1. a) Prove that outer measure of an interval is its length. **10**

b) Prove that the collection M of measurable sets is a $\sigma-$algebra. **10**

OR

c) If $E \subset [0,1]$ is a measurable set. Then prove that for each $Y \in [0,1]$ the set $E \dotplus y$ is measurable and $m(E \dotplus y) = mE$. **10**

d) Let C be a constant and f and g be two measurable real-valued functions defined on the same domain. Then prove that the functions $f+c, cf, f+g, g-f$ and fg are also measurable. **10**

UNIT - II

2. a) If ϕ and ψ are simple functions which vanish outside a set of finite measure. Then show that $\int(a\phi + b\psi) = a\int\phi + b\int\psi$ and if $\phi \geq \psi$ a.e., then prove that $\int\phi \geq \int\psi$. **10**

b) Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n(x)| \leq M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each x in E, prove that $\int_E f = \lim \int_E f_n$. **10**

OR

c) Let g be integrable over E and let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all x in E we have $f(x) = \lim f_n(x)$. Then prove that $\int_E f = \lim \int_E f_n$. **10**

d) Prove that, if $\langle f_n \rangle$ is a sequence of measurable functions that converges in measure to f. Then there is a subsequence $\langle f_{n_k} \rangle$ that converges to f almost everywhere. **10**

UNIT - III

3. a) Prove that if E is a set of finite outer measure and I a collection of intervals that cover E in the sense of vitali. Then, given E > 0, there is a finite disjoint collection $\{I_1, I_2, ...I_N\}$ of intervals in I such that $m^*\left[E \sim \bigcup_{n=1}^{N} I_n\right] < \in$. **10**

b) Prove that : A function f is of bounded variation on [a, b] if and only if the difference of two monotone real-valued functions on [a, b], **10**

OR

c) Prove that if f is bounded and measurable on [a, b] and **10**

$F(x)=\int_a^x F(t)\,dt+F(a)$, then $F'(x)=f(x)$ for almost all x in [a, b].

d) If ϕ is a continuous function on (a, b) and if one derivative (say D^+) of ϕ is nondecreasing, then prove that ϕ is convex. **10**

UNIT – IV

4. a) Prove that the space L^∞ is a normed linear space with the norm $\|f\|_\infty = \text{ess sup}\,|f(t)|$. **10**

b) State and prove Minkowski inequality for $1 \le P \le \infty$. **10**

OR

c) Prove that every convergent sequence is a Cauchy sequence. **10**

d) State and prove Riesz Representation theorem. **10**

5. a) Show that if A is countable, then $m^*A = 0$. **5**

b) Let f be a bounded function defined on [a, b]. If f is Riemann integrable on [a, b], then prove that it is measurable and $R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$. **5**

c) Prove that if f is integrable on [a, b], then the function F defined by $F(x)=\int_a^x f(t)\,dt$ is a continuous function of bounded variation on [a, b]. **5**

d) Let $1 \le P < \infty$. Then prove that for a, b, t nonnegative we have $(a+tb)^p \ge a^p + ptba^{p-1}$. **5**

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