M.Sc.(Mathematics) (NEP Pattern) Semester-I

NEP-61 - DSC-1 - Advanced Abstract Algebra

P. Pages : 2 **GUG/W/24/15112**

Time : Three Hours * 5 9 9 5 * Max. Marks : 80

Notes : 1. Solve all five questions.
2. All questions carry equal marks.

1. a) Let N be a subgroup of a group G. Then prove that the following are equivalent: **8**

i) $N \lhd G$.

ii) $xNx^{-1} = N$ for every $x \in G$

iii) $xN = Nx$ for every $x \in G$

iv) $(xN)(yN) = xyN$ for every $x, y \in G$

b) State and prove third isomorphism theorem. **8**

OR

c) Let G be a group acting on a set X. Then prove that the set of all orbits in X under G is a partition of X. Also prove that for any $x \in X$ there is a bijection $Gx \to G/G_x$ and hence $|Gx| = [G : G_x]$. **8**

d) Prove that the set $Aut(G)$ of all automorphisms of a group G is a group under composition of mappings, and $In(G) \lhd Aut(G)$. Moreover, show that $G/Z(G) \cong In(G)$. **8**

2. a) Prove that: A group G is solvable group if and only if G has a normal series with abelian factors. Further, a finite group is solvable if and only if its composition factors are cyclic groups of prime orders. **8**

b) Prove that: If G be a nilpotent group, then every subgroup of G and every homomorphic image of G are nilpotent. **8**

OR

c) Prove that: If A_n is a normal subgroup of S_n and $n > 1$, then A_n is of index 2 in S_n and $|A_n| = n!/2$. **8**

d) Prove that any permutation $\sigma \in S_n$ is a product of pairwise disjoint cycles. This cyclic factorization is unique except for the order in which the cycles are written and the inclusion or omission of cycles of length 1. **8**

3. a) Let A be a finite abelian group and let P be a prime. If p divides $|A|$, then prove that A has an element of order p. **8**

b) Let $H_1, H_2, ..., H_n$ be a family of subgroups of a group G, and let $H = H_1 H_2 ... H_n$. Then prove that the following are equivalent: **8**

i) $H_1 \times H_2 \times .... \times H_n \cong H$ under the cannonball mapping that sends $(x_1, x_2, ..., x_n)$ to $x_1 x_2 ... x_n$.

ii) $H_i \triangleleft H$, and every element $x \in H$ can be uniquely expressed as $x = x_1 x_2 .... x_n, x_i \in H_i$.

iii) $H_i \triangleleft H$ and if $x_1 x_2 .... x_n = e$, then each $x_i = e$.

iv) $H_i \triangleleft H$ and $H_i \cap (H_1 .... H_{i-1} H_{i+1} .... H_n) = \{e\}, 1 \le i \le n$.

OR

c) Prove that there are no simple groups of orders 56 and 36. **8**

d) Show that, if a group of order p^n contains exactly one subgroup each of orders $p, p^2, ..., p^{n-1}$, then it is cyclic. **8**

4. a) For any ring R and any ideal $A \neq R$, then prove that the following are equivalent: **8**

i) A is maximal

ii) The quotient ring R/A has no nontrivial ideals.

iii) For any element $x \in R, x \notin A, A + (x) = R$.

b) Prove that: If a ring R has unity, then every ideal I in the matrix ring R_n is of the form A_n, where A is some ideal in R. **8**

OR

c) Let R be a commutative principal ideal domain with identity. Then prove that any non-zero ideal $P \neq R$ is prime if and only if it is maximal. **8**

d) Let R be a commutative ring with unity in which each ideal is prime, then show that R is a field. **8**

5. a) Let H and K be distinct maximal normal subgroup of G. Then prove that $H \cap K$ is a maximal normal subgroup of H and also of K. **4**

b) Show that the Jordan-Holder theorem implies the fundamental theorem of arithmetic. **4**

c) Prove that: If G is a cyclic group of order m, n, where (m, n) = 1, then $G \cong H \times K$, where H is a subgroup of order m and K is a subgroup of order n. **4**

d) Show that if D is a division ring, then $R = D_n$ has no nontrivial ideals. **4**

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