M.Sc. - II (Mathematics) (NEP Pattern) - Semester-III

03NEPMATH01 - Major - Complex Analysis

P. Pages : 3 **GUG/W/24/16013**

Time : Three Hours *7317* Max. Marks : 80

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. A) For a given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ define the number R, $0 \le R \le \infty$, by $\frac{1}{R} = \lim\sup |a_n|^{1/n}$, then prove that **8**

a) If $|z-a| < R$, the series converges absolutely.

b) If $|z-a| > R$, the terms of the series become unbounded and so the series diverges;

c) If $0 < r < R$ then the series converges uniformly on $\{z; |z| \le r\}$.

B) Let f & g be analytic on G & Ω respectively and suppose $F(G) C\ \Omega$. Then prove that g.f is analytic on G and $(g \circ f)'(z) = g'(f(z)) f'(z)$ for all z in G. **8**

OR

C) Let u & v be real valued functions defined on a region G and suppose that u & v have continuous partial derivatives. Then prove that $f : G \to \mathbb{C}$ defined by f(z) = u(z) + iv(z) is analytic iff u & v satisfy the Cauchy-Riemann equations. **8**

D) If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $F'(z) = 0$ for all z in G, then prove that F is constant. **8**

UNIT – II

2. A) Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Then prove that (z_1, z_2, z_3, z_4) is a real number iff all four points lie on a circle. **8**

B) Let $f : G \to \mathbb{C}$ be analytic & suppose $\bar{B}(a;r) \subset G (r > 0)$. If $\gamma(t) = a + r e^{it}, 0 \le t \le 2\pi$, then prove that $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw$ for $|z-a| < \gamma$. **8**

OR

C) Let f be analytic in B(a;R); then prove that $f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n$ for $|z-a|<R$ where $a_n = \frac{1}{n!} f^{(n)}(a)$ & this series has radius of convergence $\geq R$. **8**

D) If $\gamma:[0,1]\to \mathbb{C}$ is a closed rectifiable curve & $a \notin \{\gamma\}$ then prove that $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is an integer. **8**

UNIT – III

3. A) Let G be an open subset of the plane and $F: G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w)=0$ for all w in $\mathbb{C}-G$, then for a in $G-\{\gamma\}$, prove that $n(\gamma;a) f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$ **8**

B) Show that **8**

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

OR

C) Let G be a region and suppose that f is a nonconstant analytic function on G. Then prove that for any open set U in G, F(U) is open. **8**

D) Let F be meromorphic in G with poles $p_1, p_2, \ldots\ldots p_m$ and zeros $z_1, z_2, \ldots\ldots z_n$ counted according to multiplicity. If γ is a closed rectifiable curve in G with $\gamma \approx 0$ and not passing through $p_1, p_2, \ldots\ldots, p_m$; $z_1, z_2, \ldots\ldots, z_n$; then prove that **8**

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j)$$

UNIT – IV

4. A) Let G be a region in $\mathbb{C}$ and F an analytic function on G. Suppose there is a constant M such that $\limsup_{z\to a} |f(z)| \leq M$ for all a in $\partial_\infty G$. Then prove that $|f(z)| \leq M$ for all Z in G. **8**

B) Let G be a simply connected region and let F be an analytic function on G. Suppose there is an analytic function $\phi: G \to \mathbb{C}$ which never vanishes and is bounded on G. If M is a constant & $\partial_\infty G = A \cup B$ such that: **8**

a) for every a in A, $\limsup_{z\to a} |f(z)| \leq M$;

b) for every b in B, and n > 0, $\limsup_{z\to b} |f(z)|\;|\phi(z)|^n \leq M$; then prove that $|f(z)| \leq M$ for all Z in G.

OR

C) Prove that a differentiable function f on [a, b] is convex iff f' is increasing. **8**

D) Let $a \geq \frac{1}{2}$, $G = \left\{ z : |\arg z| < \frac{\pi}{2a} \right\}$, and suppose that for every w in ∂G, **8**

$\lim_{z \to w} \sup |f(z)| \leq M$. More over, assume that for every $\partial > 0$ there is a constant P(which may depend on δ) such that $|f(z)| \leq P \exp\left(\delta |z|^a\right)$ for Z in G and |z| sufficiently large.

Then prove that $|f(z)| \leq M$ for all Z in G.

5. A) Find the radius of convergence for the power series $\sum_{n=0}^{\infty} Z^{n!}$ **4**

B) Define :- **4**

i) Winding number

ii) Zero of analytic function

C) Define :- **4**

i) Isolated singularity

ii) Removable singularity

D) State Schwarz's Lemma **4**

****