M.Sc.(Mathematics) (NEP Pattern) Semester-I

NEP-64-2 / DSE2 - Real Analysis-I

P. Pages : 3 **GUG/W/24/15116**

Time : Three Hours Max. Marks : 80

Notes :
1. Solve all five questions.
2. All questions carry equal marks.

UNIT – I

1. a) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on a compact metric space $< M, \rho >$ such that $f_1(x) \le f_2(x) \le ... \le f_n(x) \le ...$ $(x \in M)$. If $\{f_n\}_{n=1}^{\infty}$ converges (point wise) on M to the continuous function f, then prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on M. **8**

b) Prove that: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of function in $R[a,b]$, and if $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on [a, b], then f is also in R[a, b]. **8**

OR

c) Prove that: If the power series $\sum_{k=0}^{\infty} a_k x^k$ converges for x $x = x_0$ $(\text{where } x_0 \ne 0)$, then $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[-x_1, x_1]$ where x_1 is any number such that $0 < x_1 < |x_0|$. **8**

d) Prove that: If $\sum_{k=0}^{\infty} a_k$ converges, then $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly for $0 \le x \le 1$. **8**

UNIT – II

2. a) Let A be open in R^m;let $f : A \to R$ be a function of class C^2. Then prove that for each $a \in A$, $D_k D_j f(a) = D_j D_k f(a)$. **8**

b) Show that the function $f(x, y) = |xy|$ is differentiable at 0, but is not of class C^1 in any neighbourhood of 0. **8**

OR

c) State and prove the Mean-value theorem. **8**

d) Let $f: R^2 \to R^2$ be defined by the equation $f(x, y) = \left(e^x \cos y, e^x \sin y\right)$. **8**

i) Show that f is one-to-one on the set A consisting of all (x, y) with $0 < y < 2\pi$.

ii) What is the set $B = f(A)$?

iii) If g is the inverse function, find Dg(0, 1).

UNIT – III

3. a) Let P be a partition of the rectangle Q. Let $f: Q \to R$ be a bounded function. If P'' is a refinement of P, then prove that $L(f, P) \le L(f, P'')$ and $U(f, P'') \le U(f, P)$. **8**

b) Let Q be a rectangle. Let $f: Q \to R$ be a bounded function. Then prove that $\underline{\int}_Q f \le \overline{\int}_Q f$ and equality holds if and only if given $\varepsilon > 0$, there exists a corresponding partition P of Q for which $U(f, P) - L(f, P) < \varepsilon$. **8**

OR

c) Let Q be a rectangle in R^n. Let $f: Q \to R$, Assume that f is integrable over Q. Then prove that. **8**

i) If f vanishes except on a set of measure zero, then $\int_Q f = 0$.

ii) If f is non-negative and if $\int_Q f = 0$, then f vanishes except on a set of measure zero.

d) Prove that if S is a simple region in R^n, then S is compact and rectifiable. **8**

UNIT – IV

4. a) Let Q be a rectangle in R^n. Prove that there is a C^∞ function $\phi: R^n \to R$ such that $\phi(x) > 0$ for $x \in \text{Int } Q$ and $\phi(x) = 0$ otherwise. **8**

b) Let A be open in R^n. Let $f: A \to R$ be continuous. If f vanishes outside the compact subset C of A, then prove that the integrals $\int_A f$ and $\int_C f$ exist and are equal. **8**

OR

c) Let $I = [a, b]$. Let $g: I \to R$ be a function of class C^1, with $g'(x) \ne 0$ for $x \in (a, b)$. Then the set g(I) is a closed interval J with end points g(a) and g(b). If $f: J \to R$ is continuous, then prove that $\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'$. **8**

d) Let A be an n by n matrix. Let $h : R^n \to R^n$ be a linear transformation $h(x) = A.x.$ Let S be a rectifiable set in R^n and let T = h (S). Then prove that $\nu(T) = |\det A| . \nu(S)$. **8**

5. a) Let $u_1, u_2, ...$ be real-valued functions on the metric space M and if $\sum_{n=1}^{\infty} u_n$ converges uniformly to f on M and if each u_n is continuous at the point $a \in M$, then prove that f is also continuous at a. **4**

b) Let $A \subset R^m$; let $f : A \to R^n$. If f is differentiable at a, then prove that f is continuous at a. **4**

c) Let Q be a rectangle. Let $f : Q \to R$ be a bounded function. If P and P' are any two partitions of Q, then prove that $L(f, P) \le U(f, P')$. **4**

d) State Change of variables theorem. **4**

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