M.Sc. - I (Mathematics) (New CBCS Pattern) Semester-I

PSCMTH04 - Linear Algebra

P. Pages : 2 **GUG/W/24/13740**

Time : Three Hours Max. Marks : 100

Notes :
1. Solve all the **five** questions.
2. Each question carries equal marks.

UNIT – I

1. a) Let V be a vector space and W a subset of V. Then prove that W is a subspace of V if and only if the following three conditions hold for the operations defined in V. **10**

i) $0 \in W$

ii) $x + y \in W$, whenever $x \in W$ and $y \in W$

iii) $cx \in W$, whenever $c \in F$ and $x \in W$.

b) Prove that the span of any subset S of a vector space V is a subspace of V. Moreover, prove that any subspace of V that contains S must also contain the span of S. **10**

OR

c) State and prove Replacement theorem. **10**

d) Let S be a linearly independent subset of a vector space V. Then prove that there exists a maximal linearly independent subset of V that contains S. **10**

UNIT-II

2. a) Let V and W be vector spaces and let $T : V \to W$ be linear. Then prove that N(T) and R(T) are subspaces of V and W respectively. **10**

b) Let V and W be vector spaces and let $T : V \to W$ be linear. If V is finite dimensional then prove that nullity (T) + rank(T) = dim V. **10**

OR

c) Let V and W be finite-dimensional vector spaces (over the same field). Then prove that V is isomorphic to W if and only if dim V = dim W. **10**

d) Let V and W be finite-dimensional vector spaces with ordered bases β and γ respectively. Let $T : V \to W$ be linear. Then prove that T is invertible if and only if $[T]_{\beta}^{\gamma}$ is invertible. Furthermore, prove that $\left[T^{-1}\right]_{\gamma}^{\beta} = \left(\left[T\right]_{\beta}^{\gamma}\right)^{-1}$. **10**

UNIT – III

3. a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. **10**

b) Let T be a linear operator on a vector space V and let $\lambda_1, \lambda_2, ..., \lambda_k$ be distinct eigenvalues of T. If $v_1, v_2, ..., v_k$ are eigenvector of T such that λ_i corresponds to $v_i (1 \le i \le k)$ then prove that $\{v_1, v_2, ..., v_k\}$ is linearly independent. **10**

OR

c) Prove that a linear operator T on a finite dimensional vector space V is diagonalizable if and only if V is the direct sum of the eigen spaces of T. **10**

d) Let T be a linear operator on a finite dimensional vector space V and let W be T-invariant subspace of V. Then prove that the characteristic polynomial of T_W divides the characteristic polynomial of T. **10**

UNIT – IV

4. a) Let V be ann inner product space and $S = \{w_1, w_2, ...w_n\}$ be a linearly Independent subset of V. Define $S' = \{v_1, v_2, ...v_n\}$ where $v_1 = w_1$ and $v_k = w_k - \Sigma_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j$, for $2 \le k \le n$. Then proves that S' is an orthogonal set of non-zero vectors such that span $(S') = \text{span}(S)$. **10**

b) Let V be a finite dimensional inner product space and let T be a linear operator on V. Then prove that there exists a unique function $T^*: V \to V$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$, for all $x, y \in V$. Furthermore, show that T* is linear. **10**

OR

c) Let V be a finite-dimensional inner product space over F, and let $g: V \to F$ be a linear transformation, Prove that there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$, for all $x \in V$. **10**

d) Let T be a linear operator on a finite dimensional inner product space of V. suppose that the characteristic polynomial of T splits. Then prove that there exists an orthonormal basis β of V such that the matrix $[T]_\beta$ is upper triangular. **10**

5. a) Prove that any intersection of subspace of a vector space V is a subspace of V. **5**

b) Let V and W be vector spaces and let $T: V \to W$ be linear. Then prove that T is one-one if and only if $N(T) = \{0\}$. **5**

c) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_{2\times 2}(R)$ then prove that A is diagonalizable. **5**

d) Orthonormalize the set $\{w_1, w_2, w_3\}$, where $w_1 = (1,0,1,0), w_2 = (1,1,1,1), w_3 = (0,1,2,1)$ **5**

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