M.Sc. (Mathematics) (NEP Pattern) Semester-II

Major DSC-2 - Measure Theory

P. Pages : 2 **GUG/W/24/15394**

Time : Three Hours Max. Marks : 80

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

1. a) If $\{A_n\}$ is a countable collection of set of real numbers, then prove that $m^*(\bigcup A_n) \le \sum m^* A_n$. **8**

b) Prove that the interval (a, ∞) is measurable. **8**

OR

c) If $\{E_n\}$ is an infinite decreasing sequence of measurable sets, such that $m E_1$ is finite, then prove that $m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m E_n$. **8**

d) Let c be constant and f and g two measurable real-valued functions defined on the same domain. Then prove that the functions $f + c, cf, f + g, g - f$ and $f g$ are also measurable. **8**

2. a) Prove that if ϕ and ψ are two simple functions, which vanish outside a set of finite measure then prove that $\int(a\phi + b\psi) = a\int\phi + b\int\psi$ and if $\phi \ge \psi$ a.e., then show that $\int\phi \ge \int\psi$. **8**

b) State and prove the Bounded convergence theorem. **8**

OR

c) State and prove Monotone convergence theorem. **8**

d) Let f and g be integrable over E. Then prove that **8**

i) The function cf is integrable over E, and $\int_E cf = c\int_E f$.

ii) The function f+g is integrable over E, and $\int_E f + g = \int_E f + \int_E g$.

iii) If $f \le g$ a.e., then $\int_E f \le \int_E g$.

iv) If A and B are disjoint measurable sets contained in E, then $\int_{A\cup B} f = \int_A f + \int_B f$.

3. a) Let f be an integrable function on [a, b], and if $F(x)=\int_a^x f(t)\,dt+F(a)$ then prove that $F'(x)=f(x)$ for almost all x in [a, b]. **8**

b) State and prove Vitali's lemma. **8**

OR

c) Prove that a function F is an indefinite integral if and only if it is absolutely continuous. **8**

d) If f is absolutely continuous on [a, b] and $f'(x)=0$ a.e., the prove that f is constant. **8**

4. a) Prove that $||f+g||_1 \le ||f||_1 + ||g||_1$. **8**

b) Prove that the L^P spaces are complete. **8**

OR

c) State and prove Minkowski inequality for 0 < p < 1. **8**

d) Prove that L^∞ is complete. **8**

5. a) Prove that the set [0, 1] is not countable. **4**

b) Let $\phi=\sum_{i=1}^{n} a_i \chi_{E_i}$, with $E_i \cap E_j = \phi$, for $i \neq j$. Suppose each set E_i is a measurable set of finite measure, then prove that $\int \phi = \sum_{i=1}^{n} a_i mE_i$. **4**

c) State and prove Jensen inequality. **4**

d) Define: **4**

i) Complete space   ii) Banach space

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