M.Sc.(Mathematics) (CBCS NEP Pattern) Semester-II

Major DSC-1 - Field theory

P. Pages : 2 **GUG/W/24/15393**

Time : Three Hours Max. Marks : 80

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT-I

1. a) Let R be a UFD, and $a, b \in R$. Then prove that there exists a greatest common divisor of a and b that is uniquely determined to within an arbitrary unit factor. **8**

b) If $f(x), g(x) \in R[x]$, where R is a UFD, then prove that $c(f g) = c(f)c(g)$. **8**

OR

c) Let R be a unique factorization domain. Then prove that the polynomial ring $R[x]$ over R is also a unique factorization domain. **8**

d) Prove that every Euclidean domain is a PID. **8**

UNIT-II

2. a) Let $p(x)$ be an irreducible polynomial in $F[x]$. Then prove that there exists an extension E of F in which $p(x')$ has a root. **8**

b) Let $F \subseteq E \subseteq K$ be fields. If $[K:E] < \infty$ and $[E:F] < \infty$, then prove that **8**

a) $[K:F] < \infty$

b) $[K:F] = [K:E]\,[E:F]$

OR

c) Let $f(x) = a_0 + a_1 x + \dots\dots + a_{n-1}x^{n-1} + x^n \in \mathbb{Z}[x]$ be a monic polynomial. If $f(x)$ has a root $a \in \mathbb{Q}$, then prove that $a \in \mathbb{Z}$ and $a \mid a_0$. **8**

d) Let $f(x) \in \mathbb{Z}[x]$ be primitive. Then prove that $f(x)$ is reducible over $\mathbb{Q}$ if and only if $f(x)$ is reducible over $\mathbb{Z}$. **8**

UNIT-III

3. a) Prove that the prime field of a field F is either isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}/(p), p$ prime. **8**

b) Let E be an extension of a field F, and let $\alpha \in E$ be algebraic over F. Then prove that α is separable over F if and only if $F(\alpha)$ is a separable extension of F. **8**

OR

c) Prove that the multiplicative group of nonzero elements of a finite field is cyclic. **8**

d) If E is a finite separable extension of a field F, then prove that E is a simple extension of F. **8**

UNIT-IV

4. a) Let $E = \mathbb{Q}\left(\sqrt[3]{2}, \omega\right)$, where $\omega^3 = 1,\ \omega \neq 1$. Let σ_1 be the identity automorphism of E, and let σ_2 be an automorphism of E such that $\sigma_2(\omega) = \omega^2$ and $\sigma_2\left(\sqrt[3]{2}\right) = \omega\left(\sqrt[3]{2}\right)$. If $G = \{\sigma_1, \sigma_2\}$ then prove that $E_G = \mathbb{Q}\left(\sqrt[3]{2}\ \omega^2\right)$. **8**

b) Prove that the Galois group of $x^4 - 2 \in \mathbb{Q}[x]$ is the octic group (= group of symmetries of square). **8**

OR

c) Prove that the group $G(\mathbb{Q}(\alpha)/\mathbb{Q})$, where $\alpha^5 = 1$ and $\alpha \neq 1$, is isomorphic to the cyclic group of order 4. **8**

d) Prove that every polynomial $f(x) \in \mathbb{C}[x]$ factors into linear factors in $\mathbb{C}[x]$. **8**

5. Solve all the four questions.

a) Define (i) prime element (ii) irreducible element. **4**

b) Show that $x^3 + 3x + 2 \in \mathbb{Z}/(7)[x]$ is irreducible over the field $\mathbb{Z}/(7)$. **4**

c) Define (i) Splitting field (ii) Prime field. **4**

d) Define (i) Galois group (ii) Galois extension. **4**

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